MGSC 1205

Quantitative Methods I

Slides Eight – Sinking Fund Problem, Integer Programming & Binary Variables

Ammar Sarhan

Shadow Price

- The shadow price is the change in the objective function value for a one-unit increase in a constraint's RHS value.
- The value of the shadow price can provide decision makers powerful insight into problems
  - you have a constraint that limits the amount of labor available to 80 hours per week, the shadow price will tell you how much you would be willing to pay for an additional hour of labor

Sensitivity Analysis

- Sensitivity analysis is used to determine effects on the optimal solution within specified ranges for the objective function coefficients (*OFCs),* and right hand side (*RHS*) values .

- Basic Question: How does our solution change as the input parameters change?
  - **Allowable Range for *OFCs***
  - **Shadow prices**
  - **Allowable Range for *RHS* values**

Sinking Fund Portfolio

- Investor seeks to establish investment portfolio using ***least*** possible initial investment that will generate specific amounts of capital at specific time periods in future.
- Multiple period application
  - At the start of year 1, the entire initial investment is available for investing in the choices. However, in subsequent years, only the amount maturing from the prior investment is available for investment.
  - Amount used for investment at the start of a given year

    \+ Amount paid at start of this year =

    Amount maturing at the end of the previous year
  - **Balance equation at each period:**

    Cash in - Cash out = Cash in Hand (outflow)

Example: Larry's plan for daughter's college expenses

Financial needs as of : Larry anticipates that his financial need at the start of each of the following years is as follows

Year 3	$20,000
Year 4	$22,000
Year 5	$24,000
Year 6	$26,000

Investment Choices for Larry :

Choice	ROI	Maturity
A	5%	1 year
B	13%	2 years
C	28%	3 years
D	40%	4 years

Each choice is available for investment at the start of every year.
Due to risky cause, Larry wants no more than 20% of his total investment in choices C & D at any point in time.

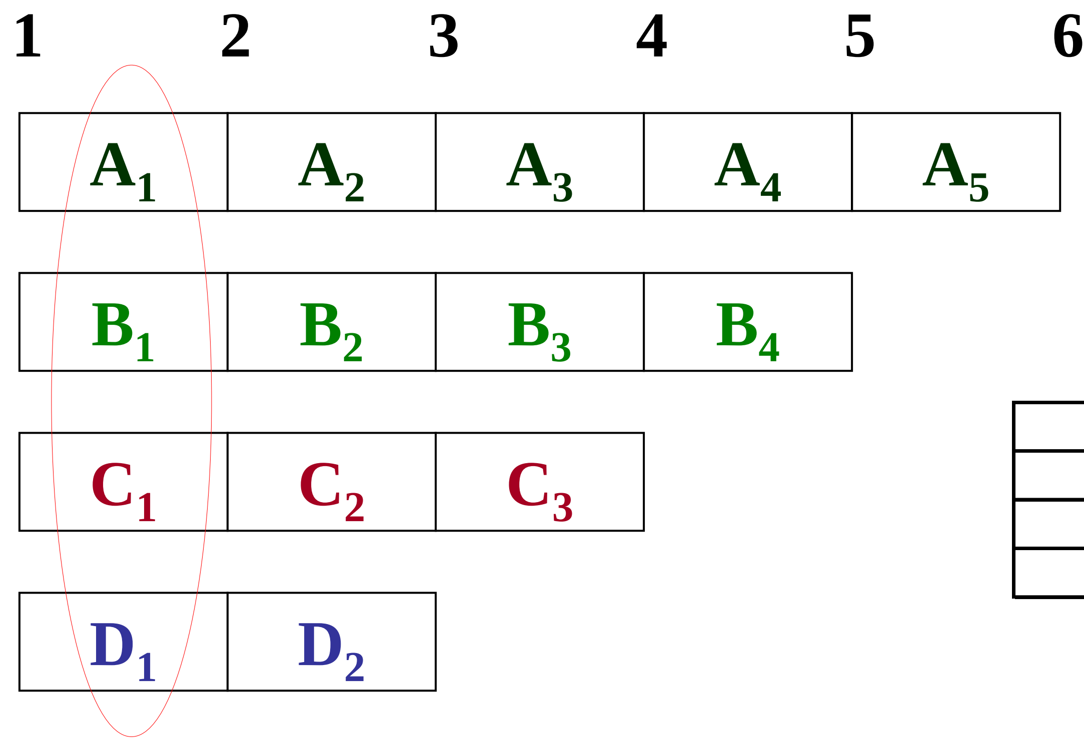
Objective: minimize the initial investment

Note: At start of year 1, the entire initial investment is available for investing in the choices. However, in subsequent years, only the amount maturing from a prior investment is available for investment.

Decision Variables

A_1 = \$ amount invested in choice A at start of year 1
B_1 = \$ amount invested in choice B at start of year 1
C_1 = \$ amount invested in choice C at start of year 1
D_1 = \$ amount invested in choice D at start of year 1
A_2 = \$ amount invested in choice A at start of year 2
B_2 = \$ amount invested in choice B at start of year 2
C_2 = \$ amount invested in choice C at start of year 2
D_2 = \$ amount invested in choice D at start of year 2
A_3 = \$ amount invested in choice A at start of year 3
B_3 = \$ amount invested in choice B at start of year 3
C_3 = \$ amount invested in choice C at start of year 3
A_4 = \$ amount invested in choice A at start of year 4
B_4 = \$ amount invested in choice B at start of year 4
A_5 = \$ amount invested in choice A at start of year 5

Investment Decisions



A	5%	1 year
B	13%	2 years
C	28%	3 years
D	40%	4 years

Year 3	$20,000
Year 4	$22,000
Year 5	$24,000
Year 6	$26,000

Start of Year 1

Cash out

initial investment

$$= A_1 + B_1 + C_1 + D_1$$

Investment Decisions

1	2	3	4	5	6
A_1	A_2	A_3	A_4	A_5	
B_1	B_2	B_3	B_4		
C_1	C_2	C_3			
D_1	D_2				

A	5%	1 year
B	13%	2 years
C	28%	3 years
D	40%	4 years

Year 3	$20,000
Year 4	$22,000
Year 5	$24,000
Year 6	$26,000

Year 2

Cash in: $1.05\ A_1$ Amount maturing at the end of year 1

Cash out: $A_2+B_2+C_2+D_2$ Amount used for investment at the start of year 2

+ *Cash in Hand* + Amount paid at start of year 2

Investment Decisions

1	2	3	4	5	6
A_1	A_2	A_3	A_4	A_5	
B_1	B_2	B_3	B_4		
C_1	C_2	C_3			
D_1	D_2				

A	5%	1 year
B	13%	2 years
C	28%	3 years
D	40%	4 years

Year 3	$20,000
Year 4	$22,000
Year 5	$24,000
Year 6	$26,000

Year 3

Cash in: $1.05\,A_2 + 1.13\,B_1$ Amount maturing at the end of year 2

Cash out: $A_3+B_3+C_3+20000$ Amount used for investment at the start of year 3

+ *Cash in Hand* + Amount paid at start of year 3

Investment Decisions

1	2	3	4	5	6
A_1	A_2	A_3	A_4	A_5	
B_1	B_2	B_3	B_4		
C_1	C_2	C_3			
D_1	D_2				

A	5%	1 year
B	13%	2 years
C	28%	3 years
D	40%	4 years

Year 3	$20,000
Year 4	$22,000
Year 5	$24,000
Year 6	$26,000

Year 4

Cash in: $1.05\,A_3 + 1.13\,B_2 + 1.28\,C_1$ Amount maturing at the end of year 3

Cash out: $A_4 + B_4 + 22000$ Amount used for investment at the start of year 4

+ *Cash in Hand* + Amount paid at start of year 4

Investment Decisions

1	2	3	4	5	6
A_1	A_2	A_3	A_4	A_5	
B_1	B_2	B_3	B_4		
C_1	C_2	C_3			
D_1	D_2				

A	5%	1 year
B	13%	2 years
C	28%	3 years
D	40%	4 years

Year 3	$20,000
Year 4	$22,000
Year 5	$24,000
Year 6	$26,000

Year 5

Cash in: $1.05\,A_4 + 1.13\,B_3 + 1.28\,C_2 + 1.40\,D_1$ Amount maturing at the end of year 4

Cash out: $A_5 + 24000$ Amount used for investment at the start of year 5

+ *Cash in Hand* + Amount paid at start of year 5

Investment Decisions

1	2	3	4	5	6
A_1	A_2	A_3	A_4	A_5	
B_1	B_2	B_3	B_4		
C_1	C_2	C_3			
D_1	D_2				

A	5%	1 year
B	13%	2 years
C	28%	3 years
D	40%	4 years

Year 3	$20,000
Year 4	$22,000
Year 5	$24,000
Year 6	$26,000

Year 6

Cash in: $1.05\,A_5 + 1.13\,B_4 + 1.28\,C_3 + 1.40\,D_2$ Amount maturing at the end of year 5

Cash out: 26000 Amount used for investment at the start of year 6

+ *Cash in Hand* + Amount paid at start of year 6

The balance equation at each year

Cash in = Cash out + Cash in Hand

Year 1: *initial investment* $= A_1+B_1+C_1+D_1$

Objective: Minimize *the initial investment* $= A_1+B_1+C_1+D_1$

Year 2: $1.05\ A_1 - (A_2+B_2+C_2+D_2) = 0$

Year 3: $1.05\ A_2 + 1.13\ B_1 - (A_3+B_3+C_3) = 20000$

Year 4: $1.05\ A_3 + 1.13\ B_2 + 1.28\ C_1 - (A_4+B_4) = 22000$

Year 5: $1.05\ A_4 + 1.13\ B_3 + 1.28\ C_2 + 1.40\ D_1 - A_5 = 24000$

Year 6: $1.05\ A_5 + 1.13\ B_4 + 1.28\ C_3 + 1.40\ D_2 = 26000$

These five constraints address the cash flow issues.

They do not account for Larry's risk performance with regard to investment in C & D in any given year.

The total investment in C & D in any given year $\leq$ (no more than) 20% of total investment in all choices that year.

1 2 3 4 5 6

A_1	A_2	A_3	A_4	A_5
B_1	B_2	B_3	B_4	
C_1	C_2	C_3		
D_1	D_2			

A	5%	1 year
B	13%	2 years
C	28%	3 years
D	40%	4 years

Year 1 risk:

$$C_1 + D_1 \leq 0.20\ (A_1 + B_1 + C_1 + D_1)$$

or

$$0.8\ (C_1 + D_1) - 0.20\ (A_1 + B_1) \leq \mathbf{0}$$ **(Year 1 risk)**

Risk constraints

1	2	3	4	5	6
A_1	A_2	A_3	A_4	A_5	
B_1	B_2	B_3	B_4		
C_1	C_2	C_3			
D_1	D_2				

A	5%	1 year
B	13%	2 years
C	28%	3 years
D	40%	4 years

Investments in prior years that may have still not matured

Year 2 risk:

$$C_1+D_1 + C_2 + D_2 \leq 0.20\ (A_2 + B_2 + C_2 + D_2 + B_1 + C_1 + D_1)$$

or

$$0.8\ (C_1+D_1 +C_2+D_2) - 0.20\ (B_1 + A_2 + B_2) \leq \mathbf{0}$$ **(Year 2 risk)**

Risk constraints

1 2 3 4 5 6

A_1	A_2	A_3	A_4	A_5
B_1	B_2	B_3	B_4	
C_1	C_2	C_3		
D_1	D_2			

A	5%	1 year
B	13%	2 years
C	28%	3 years
D	40%	4 years

Investments in prior years that may have still not matured

Year 3 risk:

$$C_3 + C_2 + C_1 + D_2 + D_1 \le 0.20\,(A_3 + B_3 + C_3 + B_2 + C_2 + C_1 + D_2 + D_1)$$

or

$$0.8\,(C_1 + D_1 + C_2 + D_2 + C_3) - 0.20\,(B_2 + A_3 + B_3) \le \mathbf{0}$$ **(Year 3 risk)**

Risk constraints

1	2	3	4	5	6
A_1	A_2	A_3	A_4	A_5	
B_1	B_2	B_3	B_4		
C_1	C_2	C_3			
D_1	D_2				

A	5%	1 year
B	13%	2 years
C	28%	3 years
D	40%	4 years

Investments in prior years that may have still not matured

Year 4 risk:

$$C_3 + C_2 + D_2 + D_1 \leq 0.20\,(A_4 + B_4 + B_3 + C_3 + C_2 + D_2 + D_1)$$

or

$$0.8\,(D_1 + C_2 + D_2 + C_3) - 0.20\,(B_3 + A_4 + B_4) \leq \mathbf{0}$$ **(Year 4 risk)**

Risk constraints

1	2	3	4	5	6
A_1	A_2	A_3	A_4	A_5	
B_1	B_2	B_3	B_4		
C_1	C_2	C_3			
D_1	D_2				

A	5%	1 year
B	13%	2 years
C	28%	3 years
D	40%	4 years

Investments in prior years that may have still not matured

Year 5 risk:

$$C_3 + D_2 \leq 0.20\ (A_5 + B_4 + C_3 + D_2)$$

or

$$0.8\ (D_2 + C_3) - 0.20\ (A_5 + B_4) \leq \mathbf{0} \qquad \textbf{(Year 5 risk)}$$

The formulation of the LP

Objective: Minimize *the initial investment* $= A_1+B_1+C_1+D_1$

S.t.:

Year 2: $1.05\,A_1 - (A_2+B_2+C_2+D_2) = 0$

Year 3: $1.05\,A_2 + 1.13\,B_1 - (A_3+B_3+C_3) = 20000$

Year 4: $1.05\,A_3 + 1.13\,B_2 + 1.28\,C_1 - (A_4+B_4) = 22000$

Year 5: $1.05\,A_4 + 1.13\,B_3 + 1.28\,C_2 + 1.40\,D_1 - A_5 = 24000$

Year 6: $1.05\,A_5 + 1.13\,B_4 + 1.28\,C_3 + 1.40\,D_2 = 26000$

$0.8\,(C_1+D_1) - 0.20\,(A_1 + B_1) \le \mathbf{0}$ **(Year 1 risk)**

$0.8\,(C_1+D_1+C_2+D_2) - 0.20\,(B_1 + A_2 + B_2) \le \mathbf{0}$ **(Year 2 risk)**

$0.8\,(C_1+D_1 + C_2 + D_2+C_3) - 0.20\,(B_2 + A_3 + B_3) \le \mathbf{0}$ **(Year 3 risk)**

$0.8\,(D_1 + C_2 + D_2+C_3) - 0.20\,(B_3 + A_4 + B_4) \le \mathbf{0}$ **(Year 4 risk)**

$0.8\,(D_2+C_3) - 0.20\,(A_5 + B_4) \le \mathbf{0}$ **(Year 5 risk)**

All variables ≥ 0

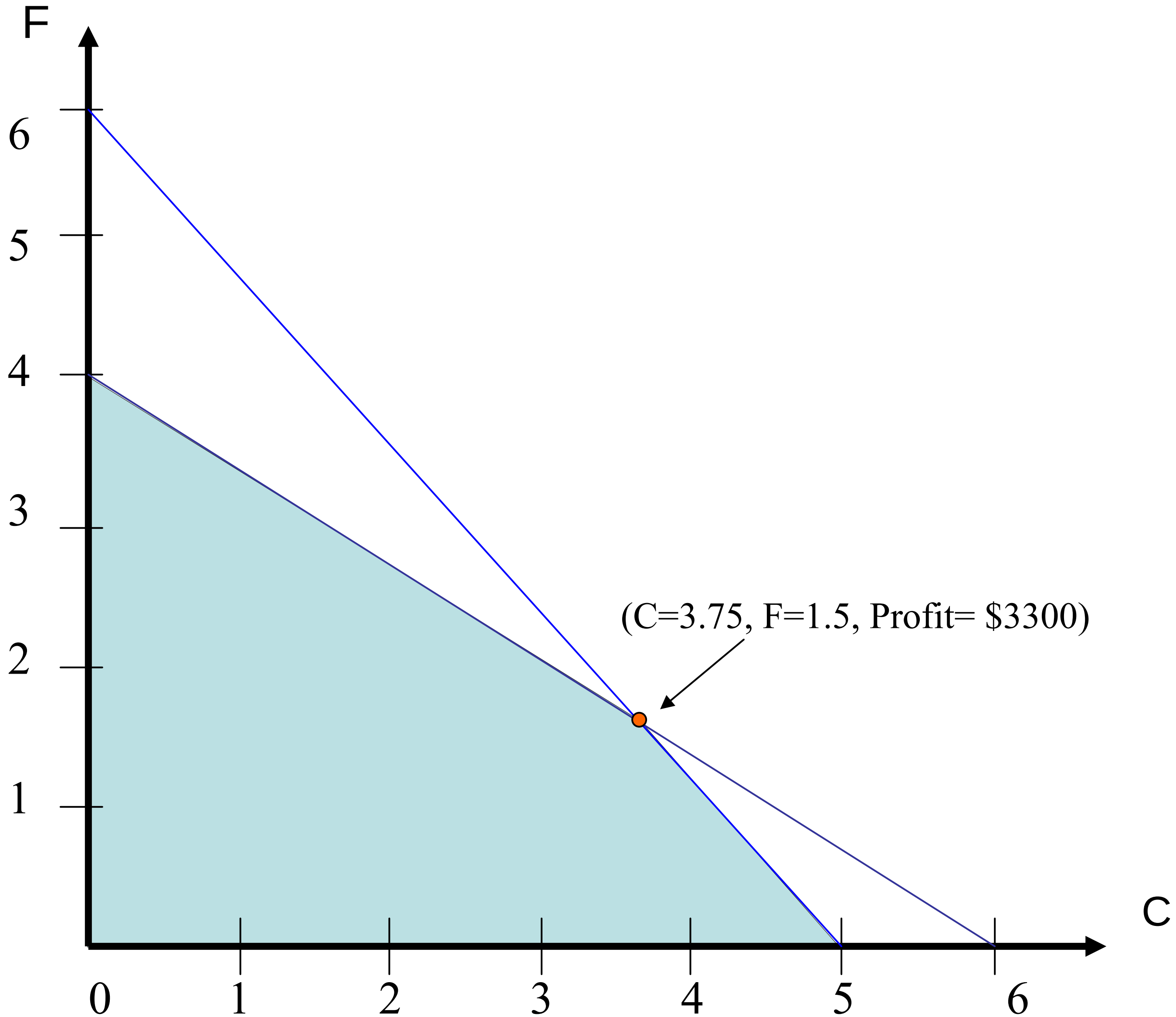
Integer Programming

- Integer Programming (IP) is the extension of LP that solves problems requiring integer solution.
  - The decision variables are not allowed to be fractional
- Two types integer variables:
  - General integer variables: can take any nonnegative integer value.
  - Binary variables: must be 0 or 1.

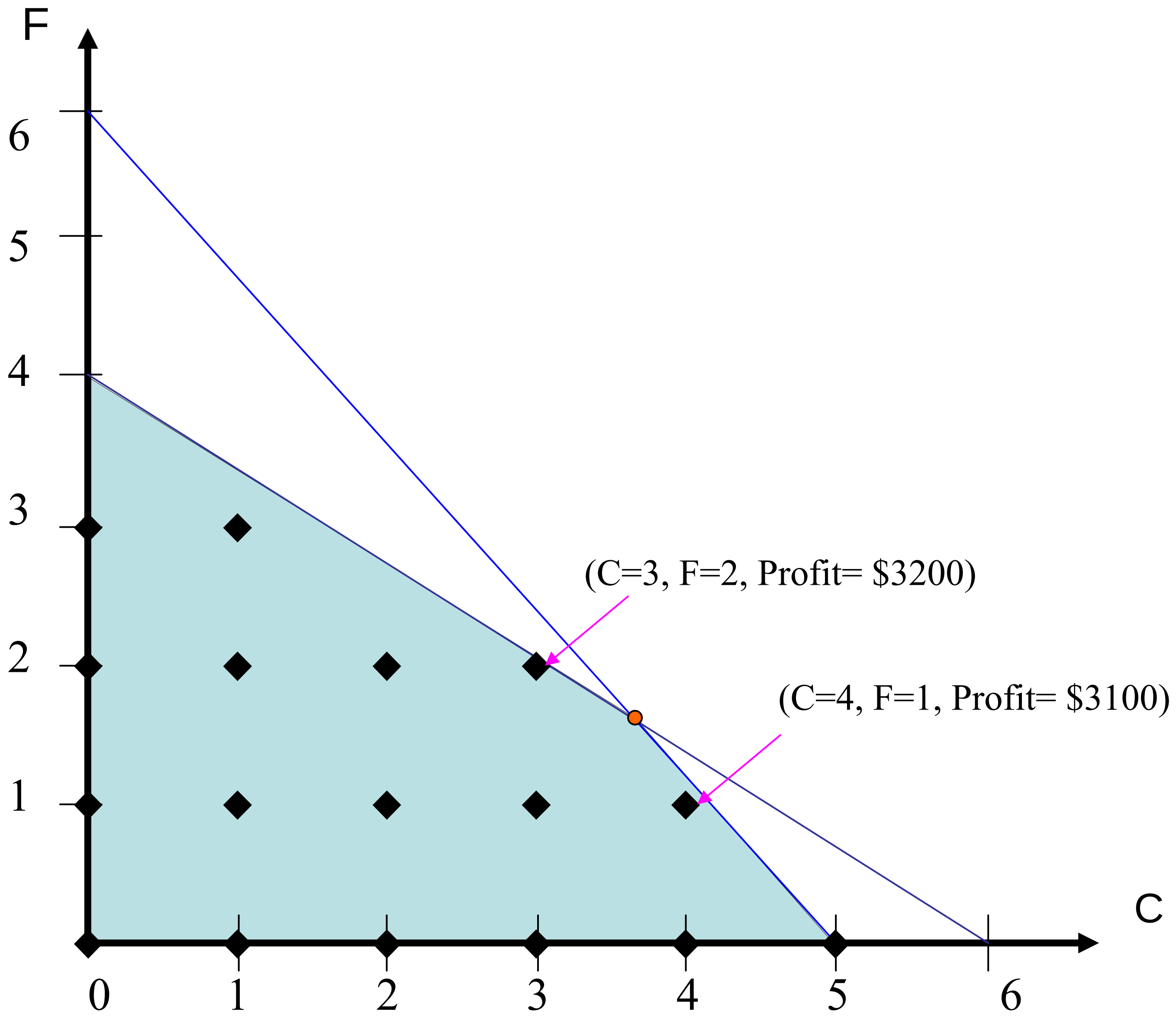
Example: Harrison Electric Company

- The company produces chandeliers & ceiling fans.
- Each chandelier produced nets $600 and each fan 700 in profit.
- Production process:
  - Each chandelier requires: 2 hours in wiring and 6 hours of assembly time
  - Each fan requires: 3 hours in wiring and 5 hours of assembly time
- Resource availability: 12 hours of wiring & 30 hours of assembly time.
- The problem is to maximize the profite.

Graphical solution



Graphical solution



Enumeration of All Integer Solutions

Chandelier (*C*)	Fans (*F*)	Profit = $600*C* + $700*F*
0	0	$0
1	0	$600
2	0	$1,200
3	0	$1,800
4	0	$2,400
0	1	$700
1	1	$1,300
2	1	$1,900
3	1	$2,500
4	1	$3,100
0	2	$1,400
1	2	$2,000
2	2	$2,600
3	2	$3,200
0	3	$2,100
1	3	$2,700
0	4	$2,800

Models With Binary Variables

Binary variables restricted to values of 0 or 1.

- Model explicitly specifies that variables are binary.
- Typical examples include decisions such as:
  - Introducing new product (introduce it or not),
  - Building new facility (build it or not),
  - Selecting team (select a specific individual or not), and
  - Investing in projects (invest in specific project or not).

Oil Portfolio Selection at SSS

SSS (Simkin, Simikin, Steinberg) specializes in recommending oil stock portfolios for wealthy clients.

Objective: maximize annul return

Company	Expected Annul Return ($1,000's)	Cost for Block of Shares($1,000's)
T	50	480
B	80	540
D	90	680
H	120	1,000
L	110	700
S	40	510
C	75	900

- At least two oil firms of *T*, *H* and *L* must be in the portfolio.
- No more than one investment can be made in *B* or *D*.
- Exactly one of *C* or *S* oil stocks must be purchased.
- If *B* stock is purchased, then *T* stock must be purchased.
- Client has $3 million available for investments

Decision Variables

- Note that the decision with regard to each company has to be one of two choices:
  - The investment firm either buys a large block of shares in the oil company or it doesn't buy the oil company's shares.

Company	Expected Annul Return ($1,000's)	Cost for Block of Shares($1,000's)
T	50	480
B	80	540
D	90	680
H	120	1,000
L	110	700
S	40	510
C	75	900

- Binary variable defined as:
  - X_i = 1: if large block of shares in company ***i*** is purchased
  - X_i = 0 : if large block of shares in company ***i*** is ***not*** purchased

  where ***i*** =**T, B, D, H, L, S, & C**
- Objective function:

Constrains

- Client has \$3 million available for investments

- Among *T*, *H* and *L*, at least two oil firms must be in the portfolio.

- No more than one investment can be made in *B* or *D*.

- Exactly one of *C* or *S* oil stocks must be purchased.

- If *B* stock is included in the portfolio, then *T* stock must also be included in the portfolio.

- For *B* and *T* stocks, the portfolio either includes both or includes neither

Models With Binary Variables

Binary variables restricted to values of 0 or 1.

- Model explicitly specifies that variables are binary.
- Typical constrains modeled using 0 or 1:
  - At least
  - At most
  - Exactly
  - If…then
  - Neither or both